1 Sequences

1.1 Definition of Sequence

A sequence is a function $f : \mathbb{R} \to \mathbb{R}$ that maps natural numbers to real numbers. Often it is convenient to define a sequence as a function $f : \mathbb{N}^+ \to \mathbb{R}$ that maps positive natural numbers to real numbers, which is the terminology we adopt in these notes.

A sequence can be:

monotonic if it is either *increasing* $(a_{n+1} \geq a_n)$ or *decreasing* $(a_{n+1} \leq a_n)$.

strictly increasing if $a_{n+1} > a_n$.

strictly decreasing if $a_{n+1} < a_n$.

1.2 Convergence

A sequence $(a_n)_{n\geq 1}$:

1. **converges to a limit** l in $\mathbb{R}$ iff *For all* $\epsilon > 0$*, we can find a* N *in* $\mathbb{N}$ *such that, for all* $n > N$*:* $|a_n - l| < \epsilon$, which can be written as $\lim_{n\to\infty} na_n = l$ or $(\mathrm{a}_n)_{n\geq 1} \to l$.
2. **converges to** ∞ in the extended real line $\mathbb{R} \cup \{\infty\}$ iff *For all* r *in* $\mathbb{R}$*, there is an* N *in* $\mathbb{N}$ *such that for all* $n \geq N$ *we have* $a_n > r$*, which can be written as* $\lim_{n\to\infty} na_n = \infty$ *or* $(a_n)_{n\geq 1} \to \infty$.
3. **converges to** $-\infty$ in the extended real line $\mathbb{R} \cup \{-\infty\}$ iff *For all* r *in* $\mathbb{R}$*, there is an* N *in* $\mathbb{N}$ *such that for all* $n \geq N$ *we have* $a_n < r$*, which can be written as* $\lim_{n\to\infty} na_n = -\infty$ *or* $(a_n)_{n\geq 1} \to -\infty$.
4. **converges** if it converges either to a real number of to $\pm\infty$.
5. **diverges** if it does not converge.

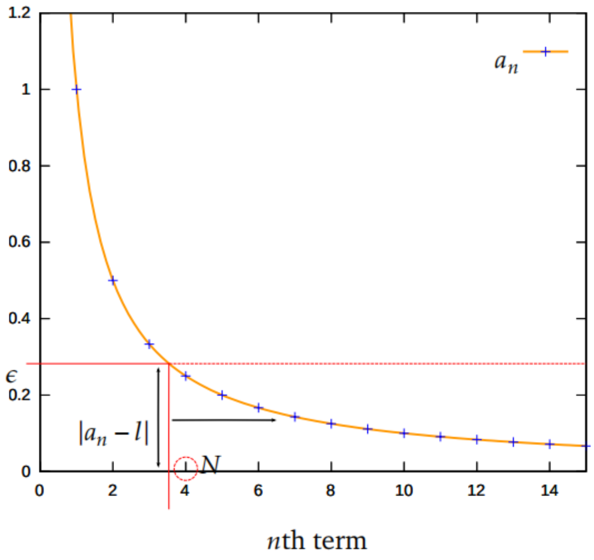

Figure 2.1: The convergence of the sequence $a_n = 1/n$ for $n \geq 1$ tending to 0

1.3 Combinations of Sequences

Given sequences $(a_n)_{n\geq 1}$ and $(a_n)_{n\geq 1}$b which converge to limits a and b respectively, and a real constant λ, then we have:

1. $\lim_{n\to\infty} \lambda a_n = \lambda a$
2. $\lim_{n\to\infty} a_n + b_n = a + b$
3. $\lim_{n\to\infty} a_n - b_n = a - b$
4. $\lim_{n\to\infty} a_n b_n = ab$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, provided that $b \neq 0$

1.4 Cauchy Sequences

A sequence $(a_n)_{n\geq 1}$ is a **Cauchy Sequence** iff for all $\epsilon > 0$, there is some N in $\mathbb{N}$ such that for all $n, m > N$ we have $|a_n - a_m| < \epsilon$.
Every sequence that converges to a real number is a Cauchy sequence.

1.4.1 Cauchy complete

A subset $A \subset \mathbb{R}$ is said to be complete if all Cauchy sequences in A converges to a limit in A.

Theorem $\mathbb{R}$ is complete.

Rational numbers $\mathbb{Q}$ are not complete. *Because* $a_{n+1} = \frac{1}{2} \cdot \left(a_n + \frac{2}{a_n}\right)$ *converges to* $\sqrt{2}$

1.5 Sandwich Theorem

Let $(u_n)_{n\geq 1}$ $(l_n)_{n\geq 1}$ be sequences, and l a real number where both $\lim_{n\to\infty} u_n = l$ and $\lim_{n\to\infty} l_n = l$ Suppose that for a third sequence $(a_n)_{n\geq 1}$a we have:

$$\textit{there is some } N \in \mathbb{N} \textit{ such that } l_n \leq a_n \leq u_n \textit{ for all } n \geq N$$

Then $\lim_{n\to\infty} a_n = l$ as well.

1.6 Ratio Tests for Sequences

Ratio Convergence Test can be used to determine whether a sequence converges to zero or diverges. It can also be applied to sequence like $a_n - l$ to prove that a_n converges to a limit l.

Ratio Convergence Test Let c in $\mathbb{R}$ be such that $0 \leq c < 1$. Suppose that there is some N in $\mathbb{N}$ such that for all $n \geq N$ we have $\left|\frac{a_n+1}{a_n}\right| \leq c$. Then $\lim_{n\to\infty} a_n = 0$.

Limit Ratio Test Suppose that the limit

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists, If $r < 1$, then the sequence $(a_n)_{n\geq 1}$ converges to 0.

1.6.1 Example: Convergence Ratio Test

Consider the sequence $(a_n)_{n\geq 1}$ where a_n equals 3^{-n} for all $n \geq 1$. Then we have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3} < 1$$

So we can choose c to be $\frac{1}{3}$ and N to be 1 and the Ratio Convergence Test passes for these values. Therefore, a$_n has limit 0$.

1.6.2 Example: Ratio Convergence Test and Limit Ratio Test

Consider the sequence $(a_n)_{n\geq 1}$ where a_n equals $\frac{1}{n!}$ for all $n \geq 1$. Then we have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1}$$

We can choose $c = \frac{1}{2}$ to apply the *Ratio Convergence Test*. We can also apply the Limit Ratio Test. Then we get

$$\lim_{n\to\infty} \left(\frac{1}{n+1}\right)_{n\geq 1} = 0$$

So the sequence has limit 0.

1.7 Subsequences of a sequence

If $f : \mathbb{N} \to \mathbb{R}$ is a sequence and $M \subset \mathbb{N}$ an infinite subset, then the restriction $f : M \to \mathbb{R}$ is called a *subsequence* of f. Using the usual notion $(a_n)_{n\geq 1}$ for a sequence, any subsequence of this sequence would be of the form $(a_{ni})_{i\geq 1}$ where n_i are positive integers with $n_1 < n_2 < n_3 \ldots$, i.e., $M = n_i : i \geq 1$.

1.7.1 Example

Suppose $a_n = \frac{1}{n}$, then $a_{ni} = \frac{1}{2i}$ is one of the subsequences of a_n.

Theorem Any subsequence of a convergent sequence converges to the limit of the sequence.

Proposition Any sequence of real numbers has a monotonic subsequence.

1.7.2 Peak

For any $m \geq 1$, we say that a_m is a **peak** of $(a_n)_{n\geq 1}$ if $a_m \geq a_n$ for all $n \geq m$.
A strictly increasing sequence will have no peaks while in a decreasing sequence every term is a peak.

1.8 Manipulating Absolute Values: Useful Techniques

$$|x| < a \Leftrightarrow -a < x < a$$

Abs1 $|x \cdot y| = |x| \cdot |y|$

Abs2 $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

Abs3 $|x + y| = |x| + |y|$, the ***triangle inequality***

Abs4 $|x - y| = ||x| - |y||$

1.9 Properties of Real Numbers – Fundamental Axiom of Analysis

Let X be a set of real numbers.

1. Let l and u be real numbers. Then:

   (a) u is an *upper bound* of X if $x \leq u$ for all $x \in X$

   (b) l is a *lower bound* of X if $l \leq x$ for all $x \in X$

   (c) a *least upper bound* of X is an upper bound s of X such that $s \leq u$ for all upper bounds u of X

   (d) a *greatest lower bound* of X is a lower bound i of X such that $l \leq i$ for all lower bounds l of X

2. We say that such a set X is
   (a) *bounded above* if X has an upper bound
   (b) *bounded below* if X has a lower bound
   (c) *bounded* if X has an upper and lower bound

supremum The least upper bounds are unique whenever they exist, which we denote with $\sup(X)$.

infimum In a dual sense, greatest lower bounds are unique if they exist, which we denote with $\inf(X)$.

For example, the set of positive real numbers $\{x \in \mathbb{R} | x > 0\}$ does not have any upper bounds and so also no least upper bound. But it has 0 and all negative numbers as lower bounds and 0 is its greatest lower bound.

1.9.1 Axiom of Completeness for Real Numbers

Every non-empty subset X of the real numbers $\mathbb{R}$ that is bounded above has a least upper bound.

1.9.2 Fundamental Theorem of Analysis

Let $(a_n)_{n \geq 1}$ be a sequence of real numbers that is

1. **increasing**, i.e. for all $n \geq m \geq 1$ we have $a_n \geq a_m$, and
2. **bounded above**, i.e. there is some u in $\mathbb{R}$ such that $a_n \leq u$ for all $n \geq 1$.

Then $s = \sup\{a_n | n \geq 1\}$ exists and is the limit of the sequence $(a_n)_{n \geq 1}$.

Theorem $\mathbb{R}$ is **complete**.

2 Continuous Functions

2.1 Limits of Functions

The function $f : [a, b] \to \mathbb{R}$ has a limit $l \in \mathbb{R}$ at $x_0 \in [a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that whenever $x \in [a, b]$ and $|x - x_0| < \delta$, then $|f(x) - l| < \delta$. We write this as $\lim_{x \to x0} f(x) = l$.

2.1.1 Example

Take function $f : [0, 1] \to \mathbb{R}$ with $f(x) = \sin(1/x)$ if $x! = 0$ and $f(0) = 0$. f has a limit for each $x_0 \in (0, 1]$ with $\lim_{x \to -x_0} f(x) = \sin(1/x_0)$ but f has no limit at $x_0 = 0$.

Proposition Suppose the two functions $f, g : [a, b] \to \mathbb{R}$ have limits $k \in R$ and $l \in R$ respectively at $x_0 \in [a, b]$. Then:

- $f \pm g$ has limit $k \pm l$ at x_0.
- $f \cdot g$ has limit $k \cdot l$ at x_0.
- f/g has limit k/l at x_0, if $l \neq 0$.